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A NOTE ON FREE PRESENTATIONS AND RESIDUAL NILPOTENCE

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1. Introduction

Let F be a non-cyclic free group, freely generated by a set $\{x_\lambda : \lambda \in \Lambda\}$, let $R \lhd F$, and let $G = F/R$. Necessary and sufficient conditions on G for F/R' to be residually nilpotent were given by Lichtman [12] (see also Passi [14]), building on work of a number of other authors. We shall show here that the same conditions are necessary and sufficient for F/S to be residually nilpotent, where S is an arbitrary verbal subgroup of R such that R/S is non-trivial and torsion-free nilpotent. When S is a term of the lower central series of R, our result is implicit in Gupta and Gupta [6], if Corollary 3.1 of Passi [14] is taken into account, and an important step in dealing with the general case was taken by Šmel'kin [16], who showed that F/S is residually torsion-free nilpotent if F/R is, and proved several other interesting results. The work of these authors has been very influential in this paper. Similar problems when S is replaced by certain Fox subgroups have been studied by Gupta and Passi [5] and Hurley [10]; their results are not covered by our theorem. More information about the background can be found in [8].

We recall that a group H is said to be discriminated by nilpotent groups of prime power exponent if, given any finite set $h_1, \dots, h_n$ of non-trivial elements of H, there exists a nilpotent group K of (finite) prime-power exponent and a homomorphism $\phi : H \to K$ such that $\phi(h_i) \neq 1$ for $i = 1, \dots, n$. For brevity, let $\mathfrak{D}$ denote the class of all groups that are either residually torsion-free nilpotent or discriminated by nilpotent groups of finite prime power exponent. By a theorem of Lichtman [12], $H \in \mathfrak{D}$ if and only if $\bigcap_{n=1}^{\infty} \mathfrak{h}^n = 0$, where $\mathfrak{h}$ is the augmentation ideal of $\mathbb{Z}H$. Our main result is as follows.

Theorem 1. *Let F be a non-cyclic free group, $R \lhd F$, and S be a fully invariant subgroup of R such that R/S is non-trivial, torsion-free and nilpotent. Then the following conditions are equivalent:*

(a) *F/S is residually nilpotent,*

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(b) $F/S \in \mathfrak{D}$
(c) $G = F/R \in \mathfrak{D}$.

It will be seen from the proof that (c)$\Rightarrow$(b) is true not just for the class $\mathfrak{D}$ as a whole, but for each of the two kinds of $\mathfrak{D}$-groups separately. It trivially remains true when R/S is residually torsion-free nilpotent.

By the remark before the statement of the theorem, we have

Corollary. *The powers of the augmentation ideal of $\mathbb{Z}G$ intersect trivially if and only if the same is true for $\mathbb{Z}[F/S]$.*

The implication (b)$\Rightarrow$(a) is trivial. We prove (a)$\Rightarrow$(c) by giving a short argument to show that F/R' is residually nilpotent and then quoting Lichtman's Theorem. The remaining implication, which is the substance of this paper, is proved using a type of *generalized Magnus embedding*. The Magnus embedding, in one of its versions, provides an embedding of F/R' in a semidirect product GM, where M is a free right $\mathbb{Z}G$-module with a basis in (1, 1) correspondence with the x_λ, the free generators of F. One can conveniently take M to be $\mathfrak{f}/\mathfrak{f}\mathfrak{r}$ (see Lemma 3.1). Related embeddings have been given by Gupta and Gupta [6], in which R' is replaced by an arbitrary term of the lower central series of R or certain Fox subgroups, and GM is replaced by certain matrix groups (see also [5]). Others were given by Šmel'kin [15], in which R' is replaced by suitable verbal subgroups and GM by a verbal wreath product. Ours has similarities with both of these. To state it, we need the following terminology. Let k be a commutative ring with identity and A be a k-algebra (not necessarily associative) generated by a set $\{a_\gamma : \gamma \in \Gamma\}$. Let $\mathbf{F}$ be the set of all functions $f : \Gamma \to \{0, 1, 2, \ldots\}$ of finite support, and for each $f \in \mathbf{F}$, let $A(f)$ be the k-submodule of A spanned by the monomials in the a_γ that have degree precisely $f(\gamma)$ in each a_γ. We say that A is *strongly graded* (with respect to this system of generators) if

$$A = \bigoplus_{f \in \mathbf{F}} A(f).$$

In this case we also have $A = \bigoplus_{n=0}^{\infty} A_n$, where A_n is the sum of $A(f)$ such that $\sum f(\gamma) = n$. This gives a more normal type of grading by 'total degree'. Obviously, if A is relatively free and strongly graded with respect to one basis, then it is with respect to any basis, since the automorphisms of A permute the bases transitively.

Theorem 2. *Let F, R, S and G be as in Theorem* 1, *and let c be the nilpotency class of R/S. Then there exists an associative ring U with* 1, *an action of G by ring automorphisms of U, and a G-invariant additive subgroup U_1 of U with the following properties:*
(a) *U_1 is $\mathbb{Z}G$-free,*
(b) *U is $\mathbb{Z}$-free as additive group,*

(c) *U_1 generates U (as a ring with 1), and $\mathbb{Q}U$ is strongly graded with respect to any $\mathbb{Q}$-basis of $\mathbb{Q}U_1$,*

(d) *$U_1^{c+1}=0$,*

(e) *F/S can be embedded in the semidirect product $G(1+J)$, where J is the ideal of U generated by U_1.*

We have written $\mathbb{Q}U$ for $\mathbb{Q}\otimes_{\mathbb{Z}} U$.

It will turn out that U is the quotient of the universal envelope of a certain relatively free Lie ring M by the ideal generated by the $(c+1)$-fold products of the elements of M. For clarity, we have isolated the properties of U that are important to us.

Except where otherwise explained, F, R, S and G will be as above throughout the rest of the paper.

2. Deduction of Theorem 1

We now deduce Theorem 1 from Theorem 2.

Lemma 1. *If F/S is residually nilpotent, then so is F/R'.*

A slightly more general result was proved in [8]. We reproduce the argument as it is quite short.

Proof. Write $H=F/S$ and $K=R/S$. Then K is a relatively free and torsion-free nilpotent group, of class c, say. Let $1=Z_0<Z_1<\cdots<Z_c=K$ be its upper central series. It is easy to see that Z_{c-1}/K' is characteristic in K/K' (see [13, 42.32]), whence $Z_{c-1}=K'$, as K/Z_{c-1} is non-trivial and torsion-free. Put $C_1=C_H(K)$. Then

$$\left[\bigcap_{n=1}^{\infty} C_1\gamma_n(H), K\right]\le\bigcap_{n=1}^{\infty}\gamma_n(H)=1,$$

so $\bigcap_{n=1}^{\infty} C_1\gamma_n(H)=C_1$, and H/C_1 is residually nilpotent. Writing $C_{i+1}/C_i=C_{H/C_i}(KC_i/C_i)$ $(i\ge 1)$, we similarly see that each H/C_i is residually nilpotent. We have $K\cap C_i=Z_i$, and in particular $C_{c-1}\cap K=K'$. Hence $[C_{c-1},K]\le K'$, and as K/K' is well known to be self centralizing in H/K', we find that $C_{c-1}=K'$. Hence H/K' is residually nilpotent, as claimed.

Now by Lichtman's results [12], we know that if F/R' is residually nilpotent, then $G\in\mathfrak{D}$. Thus, the implication (c)$\Rightarrow$(b) of Theorem 1 remains to be proved. Let $S=V(R)$, where V is a certain set of words. The hypothesis (*c*) on F/R amounts to saying that there exists a set $\mathbf{X}$ of normal subgroups of F/R, containing R, such that if T is any finite subset of F such that $T\cap R=\emptyset$, then there exists $X\in\mathbf{X}$ such that $T\cap X=\emptyset$. Furthermore the groups F/X $(X\in\mathbf{X})$ are either all torsion-free nilpotent, or all nilpotent groups of prime power exponent. The proof of a theorem of Dunwoody [3] goes over without modification to yield $V(R)=\bigcap_{X\in\mathbf{X}} V(X)$. The groups $X/V(X)$ are relatively free in the variety generated by R/S, so they are isomorphic to subgroups of Cartesian powers of R/S [13, 15.4] and are torsion-free nilpotent. Consequently the proof of (c)$\Rightarrow$(b) reduces to the case when G is either torsion-free nilpotent or a nilpotent p-group of finite exponent. I am indebted to Alexander Lichtman for pointing this out to me. We embed F/S in $G(1+J)$ as in Theorem 2, and then the following completes the proof.

Lemma 2. *Suppose that U is an associative ring with* 1 *acted on by a group G and containing a G-invariant sugroup U_1 such that* (a)–(d) *of Theorem* 2 *are satisfied. Let J be the ideal of U generated by U_1. Then if G is torsion-free nilpotent, $G(1+\mathbb{Q}J)$ is residually torsion-free nilpotent, and if G is a nilpotent p-group of finite exponent, $G(1+J)$ is residually a nilpotent p-group of finite exponent.*

Proof. First suppose that G is torsion-free nilpotent. Write $E_i=\mathbb{Q}U_1\mathfrak{g}^{i-1}$ $(i=1,2,\ldots)$. Then $\bigcap_{i=1}^{\infty} E_i=0$ [9, Theorem B2]. Let

$$J_n=\sum E_{i_1}E_{i_2}\cdots E_{i_r}$$

over all choices of $i_1,\ldots,i_r$ such that $\sum_{\tau=1}^{r} i_\tau\geq n$. We first claim that

$$\bigcap_{n=1}^{\infty} J_n=0 \tag{1}$$

To see this, let $\xi\in\bigcap_{n=1}^{\infty} J_n$. There exists a finite-dimensional $\mathbb{Q}$-subspace S of $\mathbb{Q}U_1$ such that ξ is a $\mathbb{Q}$-linear combination of products of elements of S. Choose n so that $S\cap E_n=0$ and choose a basis $\{u_\gamma:\gamma\in\Gamma\}$ of $\mathbb{Q}U_1$ adapted to both E_n and S. Then ξ can be written as a $\mathbb{Q}$-linear combination of monomials in those u_γ that belong to S. On the other hand, $\xi\in J_{nc}$, so ξ can be written as a $\mathbb{Q}$-linear combination of products $v_1\cdots v_r$, where $v_\tau\in E_{i_\tau}$ and $i_1+\cdots+i_r\geq nc$. Since $U_1^{c+1}=0$, we have $r\leq c$, whence $i_\tau\geq n$ for some n. Now express each v_σ as a linear combination of u_γ's. In each resulting monomial, at least one u_γ in E_n occurs. Equating the two expressions for ξ and using the strong grading, we obtain $\xi=0$. This gives (1).

Next, we see that

$$J_n\mathfrak{g}\leq J_{n+1}. \tag{2}$$

For consider any product $v=v_1\cdots v_r\in J_n$, with $v_\tau\in E_{i_\tau}$ and $i_1+\cdots+i_r\geq n$. We have, writing the action of G as right multiplication,

$$vg=v_1g\cdots v_rg=v_1(1+(g-1))\cdots v_r(1+(g-1))$$
$$\equiv v_1\cdots v_r \mod J_{n+1},$$

as $E_i\mathfrak{g}\leq E_{i+1}$.

Now $\mathbb{Q}J\cdot J_n+J_n\cdot\mathbb{Q}J\leq J_{n+1}$, as $E_1=\mathbb{Q}U_1$, and we see in the usual way from (2) that $1+\mathbb{Q}J=1+J_1\geq 1+J_2\geq\cdots$ is a series of $1+\mathbb{Q}J$ with factors central in $G(1+\mathbb{Q}J)$. Hence from (1) this group is residually torsion-free nilpotent.

Similar considerations apply when G is a nilpotent p-group of finite exponent, though we need more details from [9]. By expressing each lower central factor of G as a direct product of cyclic groups, we wrote down in [9] a certain $\mathbb{Z}$-basis $u(\mathbf{r})(\mathbf{r} \in \mathbf{S})$ of $\mathbb{Z}G$. Using this in conjunction with a $\mathbb{Z}G$-basis $\mathbf{A}$ of U_1, we write down a $\mathbb{Z}$-basis $\mathbf{B} = \{au(\mathbf{r}) : (a \in \mathbf{A}, \mathbf{r} \in \mathbf{S})\}$ of U_1. *By* (*c*), U is strongly graded with respect to this basis. For any $m \geq 1$, the images of the elements of $\mathbf{B}$ form a basis $\bar{\mathbf{B}}$ of $U_1/p^m U_1$ over $R_m = \mathbb{Z}/p^m\mathbb{Z}$, and clearly $\bar{U} = U/p^m U$ will be strongly graded with respect to this basis. Now $\bigcap_{m=1}^{\infty} p^m J = 0$ since U is $\mathbb{Z}$-free by (b), so we just have to show that $G(1+\bar{J})$ is residually a nilpotent p-group of finite exponent, where $\bar{J} = J + p^m U/p^m U$.

We also constructed in [9] a certain descending chain $E_0 = R_m G \geq E_1 \geq \cdots$ such that $\bigcap_{j=0}^{\infty} E_j = 0$ and $E_j \mathfrak{g} \leq E_{j+1}$. Each E_j is spanned over R_m by certain elements $\lambda u(\mathbf{r})$ $(\lambda \in R_m, \mathbf{r} \in \mathbf{S})$, of 'weight' $\geq j$. The exact definition of the weight need not concern us; however we need to note that the elements of R_m have weight bounded by some number N, so that if $\lambda u(\mathbf{r})$ has weight $\geq m$, then $u(\mathbf{r})$ has weight $\geq m - N$.

Let $F_i = \bar{U}_1 E_i$, and for $n \geq 1$, let

$$J_n = \sum F_{i_1} \cdots F_{i_r}$$

over all choices of $i_1, \ldots, i_r$ such that $i_1 + \cdots + i_r \geq n$.

Essentially as before, we see that $\bigcap_{n=0}^{\infty} J_n = 0$. Namely, if $\xi \in \bigcap_{n=1}^{\infty} J_n$, we choose an additive subgroup S of $\bar{U}_1$, generated by finitely many elements of $\bar{\mathbf{B}}$, such that ξ is an R_m-linear combination of products of elements of S. Choose n such that $S \cap F_n = 0$. Now $\xi \in J_{(n+N)c}$, so we can write ξ as a linear combination of products $a\lambda u(\mathbf{r})(a \in \mathbf{A}, \mathbf{r} \in \mathbf{S})$ in which the sum of the weights of the elements $\lambda u(\mathbf{r})$ exceeds $(n+N)c$. At most c factors occur and each λ has weight at most N, so some $u(\mathbf{r})$ has weight at least n. For this r, we have $au(\mathbf{r}) \in F_n$. Equating the two expressions for ξ and using the strong grading gives $\xi = 0$.

We see easily as before that each $G(1+\bar{J})/(1+J_n)$ is a nilpotent p-group of finite exponent, and since $\bigcap_{n=1}^{\infty} (1+J_n) = 1$ in $G(1+\bar{J})$, we have that this group is residually a nilpotent p-group of finite exponent, as claimed.

3. The embedding theorem

We revert to the notation of the introduction. Let $x \to \bar{x}$ be the homomorphism $F \to G$. Our embedding theorem may be viewed as a fairly natural extension of the now standard Magnus embedding, which is usually presented as an embedding into a certain group of matrices, but for our purposes is better described as follows.

Lemma 3.1. *Let A be the free right $\mathbb{Z}G$-module with basis $\{t_\lambda : \lambda \in \Lambda\}$, and let ϕ be the homomorphism from F to the semidirect product GA given by $\phi(x_\lambda) = (\bar{x}_\lambda, t_\lambda)$. Then* $\ker \phi = R'$.

Proof. We write the elements of GA as ordered pairs (g, a) $(g \in G, a \in A)$, with $(g_1, a_1)(g_2, a_2) = (g_1 g_2, a_1 g_2 + a_2)$. It is well known that $\mathfrak{f}/\mathfrak{fr}$ is free as a right $\mathbb{Z}G$-module on the elements $(1 - x_\lambda) + \mathfrak{fr}$; consequently we may take $A = \mathfrak{f}/\mathfrak{fr}$ and $t_\lambda = (1 - x_\lambda) + \mathfrak{fr}$. An elementary verification shows that $y \to (\bar{y}, (1 - y) + \mathfrak{fr})$ is a homomorphism from F to GA. Since it agrees with ϕ on the x_λ it coincides with ϕ. So we see that

$$\ker \phi = \{r \in R : 1 - r \in \mathfrak{fr}\} = R',$$

by the well-known isomorphism $\mathfrak{r}F/\mathfrak{fr} \simeq R/R'$ (see [4, Lemma 1]).

We now need a number of facts about Lie algebras and their universal enveloping algebras. Most of them seem to be quite well known, especially in the Soviet literature, but it seems clearest to develop what we need. The notation will be cumulative. Let L be the free Lie ring on the basis $\{y_\gamma : \gamma \in \Gamma\}$ and let T be its universal envelope; this is the free associative ring on the same generators. Obviously, both L and T have a strong grading in terms of these generators:

$$L = \bigoplus_{f \in \mathbf{F}} L(f); \qquad T = \bigoplus_{f \in \mathbf{F}} T(f),$$

where $\mathbf{F}$ is the set of functions $f : \Gamma \to \{0, 1, \ldots\}$ with finite support, $T(f)$ is spanned by the monomials of degree $f(\gamma)$ in y_γ, and $L(f) = T(f) \cap L$.

Let J be any ideal of L, let $M = L/J$, and let W be the universal enveloping algebra of M. The natural homomorphism $\theta : L \to M$ extends to a homomorphism, also denoted by θ, from T to W.

Lemma 3.2. Ker θ *is the ideal of* T *generated by* J.

Proof. Let $\bar{J}$ be the ideal of T generated by J. Clearly, $\bar{J} \leq \ker \theta$. On the other hand, the map $\psi : y + J \to y + \bar{J}$ $(y \in L)$ is a well-defined Lie homomorphism that extends to an associative homomorphism from W to $T/\bar{J}$. Clearly $\psi\theta$ agrees on L with the natural map $L \to L + \bar{J}/\bar{J}$; hence it must be the natural map $T \to T/\bar{J}$. Therefore $\ker \theta = \bar{J}$.

Lemma 3.3. *Suppose that* J *is fully invariant in* L *and* $M = L/J$ *is* $\mathbb{Z}$*-torsion-free. Then*

(i) $J = \bigoplus_{f \in \mathbf{F}} J \cap T(f)$ *and* $\bar{J} = \bigoplus_{f \in \mathbf{F}} \bar{J} \cap T(f)$,

(ii) M *and* W *are strongly graded.*

When (i) holds, one says that J and $\bar{J}$ are *multihomogeneous*.

Proof. Clearly (ii) follows from (i). Suppose that (i) is false. Any non-zero element of T has the form $x = x(f_1) + \cdots + x(f_t)$, where $f_1, \ldots, f_t$ are distinct elements of $\mathbf{F}$ and $0 \neq x(f_i) \in T(f_i)$. Choose x in J but not in $\bigoplus_{f \in \mathbf{F}} J \cap T(f)$, with t minimal subject to this. Then $t \geq 2$, and we can choose γ such that $r = f_1(\gamma) \neq f_2(\gamma) = s$. Consider the endomorphism ε that maps y_γ to $2y_\gamma$ and fixes the remaining y_δ. As J is fully invariant, it contains

$$\varepsilon(x) - 2^r x = (2^s - 2^r)x(f_2) + x'(f_3) + \cdots + x'(f_t),$$

where $x'(f_i) \in T(f_i)$ $(3 \leq i \leq t)$. By the minimality of t, $(2^s - 2^r)x(f_2) \in J \cap T(f_2)$. Since L/J is torsion-free, $x(f_2) \in J$. Hence $x - x(f_2) \in J$, and the minimality of t gives a contradiction.

By Lemma 3.2, $\bar{J}$ is the ideal of T generated by J, and so

$$\bar{J} = \bigoplus_{f \in \mathrm{F}} \bar{J} \cap T(f).$$

Lemma 3.4. *If J is fully invariant and $M = L/J$ is $\mathbb{Z}$-torsion-free, then M and W are $\mathbb{Z}$-free.*

Proof. See [7, Proposition 2.8].

It follows from the Poincaré-Birkhoff-Witt Theorem that M is naturally embedded in W. Now it is easy to see that $\mathbb{Q}M$ is relatively free (we may assume M finitely generated in proving that) and $\mathbb{Q}W$ is its universal enveloping algebra. Hence we can pass between any two sets of free generators of $\mathbb{Q}M$ by an automorphism of $\mathbb{Q}W$.

Corollary 3.5. *$\mathbb{Q}M$ and $\mathbb{Q}W$ are strongly graded with respect to any set of free generators of $\mathbb{Q}M$.*

Write $M_i = \sum M(f)$ over all f such that $\sum_\gamma f(\gamma) = i$, and define W_i similarly.

Corollary 3.6. (i) *$M = \bigoplus_{i=1}^{\infty} M_i$ and $W = \bigoplus_{i=0}^{\infty} W_i$.*
(ii) *If M is nilpotent of class c, then $M \cap (W_{c+1} \oplus \cdots) = 0$.*

Now we recall some of the more elementary parts of the Andreev correspondence [2] between certain varieties of groups and Lie algebras. Let $\mathfrak{X}$ be a collection of torsion-free nilpotent groups of class at most c, say, and $\mathfrak{V}$ be the variety they generate. As the free $\mathfrak{V}$-groups are subcartesian products of groups in $\mathfrak{X}$, they also are torsion-free nilpotent and of class at most c. Let H be one of them and let $\bar{H}$ be its Mal'cev completion. We can view $\bar{H}$ as a Lie algebra over $\mathbb{Q}$ by 'inverting the Campbell-Hausdorff formula' [1], [11], and from the way the group and Lie operations are related together, a map of $\bar{H}$ into itself is a group homomorphism if and only if it is a Lie homomorphism. Thus it is essentially clear that a set of free generators of the relatively free group H will generate $\bar{H}$ freely as a relatively free Lie algebra. Taking the variety of Lie algebras generated by all the $\bar{H}$, we obtain the variety $\mathfrak{W}$ corresponding to $\mathfrak{V}$ under the Andreev correspondence. Now any set of free Lie algebra generators of $\bar{H}$ will generate a group isomorphic to H. For this is true for one set, and the various sets are permuted transitively by the automorphisms of $\bar{H}$. In this way, we can recover $\mathfrak{V}$ from $\mathfrak{W}$.

Let M be the Lie subring ($\mathbb{Z}$-algebra) generated by a set $\mathbf{B}$ of free Lie generators of $\bar{H}$, and let W be its universal enveloping algebra. Clearly M is relatively free, so the previous discussion applies, and in particular, by Corollary 3.6, M embeds in $U = W/(W_{c+1} \oplus \cdots) = U_0 \oplus \cdots \oplus U_c$, where U_i is the image of W_i in U. If $b \in \mathbf{B}$, the series for $\exp c!b$ converges in U. The group generated by these elements is isomorphic to the subgroup of $\bar{H}$ they generate under the Campbell–Hausdorff operation, that is, to H.

Lemma 3.7. *With the above notation,* $\langle \exp c!b : b \in \mathbf{B} \rangle \cong H$.

Proof of Theorem 2. We apply the above considerations, taking $\mathfrak{V}$ to be the variety generated by R/S. Let H be the free $\mathfrak{V}$-group on the basis $\mathbf{B}$ consisting of all formal products $t_\lambda g (\lambda \in \Lambda, g \in G)$, and let M and U be as just described. We have an obvious action of G on $\mathbf{B}$ by right multiplication, and so we can make G act on M and U by Lie automorphisms and associative ring automorphisms respectively. Put $J = U_1 \oplus \cdots \oplus U_c$. The elements of $\mathbf{B}$ span U_1 additively (by the definition of U_1) and as they are linearly independent in M they remain so in U. Thus, U_1 is $\mathbb{Z}G$-free on the basis $\{t_\lambda : \lambda \in \Lambda\}$.

Conditions (b), (c) and (d) of the theorem follow from Lemma 3.4, Corollary 3.5 and the definition of U, respectively.

Let $E = \langle \exp c!t_\lambda g : \lambda \in \Lambda, g \in G \rangle$. Then $E \leq 1 + J$ and $E \cong H$ by Lemma 3.7. Thus the semidirect product $GE \leq G(1+J)$ is actually the $\mathfrak{V}$-wreath product $G \operatorname{wr}_{\mathfrak{V}} H_0$, where H_0 is the $\mathfrak{V}$-group $\mathfrak{V}$-freely generated by the t_λ, and the fact that F/S can be embedded in it follows from Šmel'kin's Theorem [15]. However it is easy to give a direct argument at this stage. We consider the map $\psi : F \to GE$ given by

$$\psi(x_\lambda) = (\bar{x}_\lambda, \exp c!t_\lambda).$$

Clearly, $\psi(R) \leq E$, and since $E \in \mathfrak{V}$, $\psi(S) = 1$. On the other hand, the elements of E lie in $1 + J$, and so each element of GE is uniquely of the form $(g, 1 + u_1 + u_2 + \cdots + u_c)$ with $g \in G$ and $u_i \in U_i$. The map θ sending the above element to (g, u_1) is a homomorphism from GE to the semidirect product GU_1, and $\theta\psi(x_\lambda) = (\bar{x}_\lambda, c!t_\lambda)$. This is the usual Magnus embedding, apart from the unimportant factor $c!$. Therefore, if R is free on the elements $\{r_\gamma : \gamma \in \Delta\}$, the elements $\theta\psi(r_\gamma)$ are linearly independent in U_1. Hence the elements $\psi(r_\gamma)$ are linearly independent mod E', and by a well known property of torsion-free relatively free nilpotent groups (see for instance [13, 42.35]) they $\mathfrak{V}$-freely generate a $\mathfrak{V}$-free group. Clearly, ψ must induce an isomorphism of R/S onto this group, so $\ker \psi = S$. This completes the proof.

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